

ON EQUIMULTIPLE MODULES

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ABSTRACT. We study the class of equimultiple modules. In particular, we prove several criteria for an equimultiple module to be a complete intersection and prove the openness of the equimultiple locus of an ideal module.

1. INTRODUCTION

Equimultiple ideals (i.e., analytic spread = height) have been extensively studied partly because of their connections to geometry. This notion is the algebraic formulation of the concept of equimultiple variety introduced by O. Zariski, which is of great importance in several aspects within the study of algebraic singularities. We refer to the article of J. Lipman [18] or the book [12] by M. Herrmann, S. Ikeda and U. Orbanz for a detailed explanation of these connections. On the other hand, the relevance of equimultiple ideals is also focused on a theorem by E. Böger (cf. [12, Theorem 19.6]) which is an extension to the equimultiple case of D. Rees' multiplicity criterion for primary ideals in terms of reductions of ideals [22]. Non primary equimultiple ideals may be produced, for instance, via linkage as shown by A. Corso, C. Polini and W. V. Vasconcelos in [7].

Multiplicity theory was extended by D. Buchsbaum and D. S. Rim [4] to submodules of finite colength in a free module introducing what is known by Buchsbaum-Rim multiplicity, while D. Rees introduced the theory of reductions and integral closure of modules in [23]. In this context of modules, Rees' multiplicity criterion was proven by D. Kirby and D. Rees in [15] and by S. L. Kleiman and A. Thorup in [16] and D. Katz gave the corresponding extension of Böger's theorem to equimultiple modules in [13]. Both Buchsbaum-Rim multiplicity and integral closure of modules play an important role in the work by T. Gaffney [10, 11] on the study of equisingularity conditions of isolated complete intersection singularities (ICIS), which has been an important source of motivation to pursue the study of multiplicity theory and related topics in the context of modules.

Equimultiple modules have also been defined by A. Simis, B. Ulrich and W. V. Vasconcelos in [24] as a particular class of ideal modules: The class of ideal modules behaves somehow similarly to the class of ideals and one is then able to define the analytic deviation of an ideal module, the equimultiple modules being those with analytic deviation zero. Their definition is slightly different but agrees with ours in the Cohen-Macaulay case. They also show how to produce such modules via linkage.

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The main purpose of this paper is to make a systematic approach to the properties of equimultiple modules by using the theory of reductions of modules. As application we obtain several criteria for an equimultiple module to be a complete intersection and prove the openness of the equimultiple locus of an ideal module extending to the case of modules the corresponding results in the ideal case.

Let R be a Noetherian ring with total ring of fractions Q and $E \subseteq G \simeq R^e$ an R -module having rank $e > 0$. In our context, many of the structural properties of E are reflected by the quotient G/E and by the e -th Fitting ideal $F_e(E)$, being these two sets related by $V(F_e(E)) \subseteq \text{Supp } G/E$. Moreover, in the case where $\text{grade } G/E \geq 2$, E is said to be an *ideal module*, and the inclusion is then an equality (see Theorem 3.6). The Fitting ideals play an important role in the study of this class of modules, intervening in the definition of complete intersection and equimultiple modules, cf. section 4. For this reason, we pay special attention to the relations between G/E and $F_e(E)$, cf. section 3.

Complete intersection modules (i. e. modules of the principal class) are of course equimultiple. We then prove several criteria for an equimultiple module to be a complete intersection extending to modules the corresponding ones in the ideal case. For example:

Theorem. [cf. Theorem 5.3] *Let R be a Cohen-Macaulay local ring, E a non-free finitely generated torsionfree R -module having rank $e > 0$. Suppose that E is generically a complete intersection. Then E is complete intersection if and only if E is equimultiple.*

In section 7 we also prove the openness of the complete intersection and the equimultiple locus, for ideal modules.

Theorem. [cf. Theorems 7.2, 7.4] *Let R be a Noetherian ring and $E \subsetneq G \simeq R^e$ an ideal module. Then*

- a) $U_{ci} = \{\mathfrak{p} \in \text{Supp } G/E \mid E_{\mathfrak{p}} \text{ is a complete intersection}\}$ is a (possibly empty) open subset in $\text{Supp } G/E$.
- b) $U_{eq} = \{\mathfrak{p} \in \text{Supp } G/E \mid E_{\mathfrak{p}} \text{ is equimultiple}\}$ is a non-empty open subset in $\text{Supp } G/E$.

As in the case of ideals, the notion of Rees algebra appears naturally in this context. Let R be a Noetherian ring and E finitely generated R -module that affords an embedding into a free R -module, $E \xrightarrow{f} G \xrightarrow{\sim} R^e$. For such a module, the Rees algebra $\mathcal{R}(E)$ of E is the R -subalgebra of the polynomial ring $R[t_1, \dots, t_e]$ generated by all linear forms $a_1 t_1 + \dots + a_e t_e$, where (a_1, \dots, a_e) is the image of an element of E in R^e under the embedding $\varphi \circ f$. Summarizing,

$$\mathcal{R}(E) := \bigoplus_{n \geq 0} \mathcal{S}(f)_n(\mathcal{S}(E)_n) \subseteq R[t_1, \dots, t_e],$$

where $\mathcal{S}(f): \mathcal{S}(E) \rightarrow \mathcal{S}(G) = R[t_1, \dots, t_e]$ is the induced map of symmetric algebras. One should note that, for given different embeddings of E into free R -modules, we can get non isomorphic Rees algebras, see for instance A. Micali [20, Chapitre III, 2.

Un example] or the more recent D. Eisenbud, C. Huneke and B. Ulrich [9, Example 1.1]. See also these papers for a discussion about the uniqueness of the definition of the Rees algebra of a module.

In the particular case that E is a finitely generated torsionfree R -module with rank e , then E affords an embedding into a free module of the same rank, $E \xrightarrow{f} G \xrightarrow{\varphi} R^e$ and one can see (because E is torsion free) that

$$\mathcal{R}(E) \simeq \mathcal{S}(E)/\tau_R(\mathcal{S}(E)),$$

so the Rees algebra of E is independent of the embedding f . We then denote by E^n the n -th graded piece of $\mathcal{R}(E)$, that is $E^n := \mathcal{R}(E)_n$ and call it the n -th Rees power of E .

A special case is the Rees algebra of a module $E = I_1 \oplus \cdots \oplus I_e$ where I_1, \dots, I_e are R -ideals. Then, $\mathcal{R}(E)$ is the multi-Rees algebra $\mathcal{R}(I_1, \dots, I_e) = R[I_1 t_1, \dots, I_e t_e]$. In section 6 we give some examples of equimultiple modules of this type. Finally, in section 8, we characterize the non-free locus of the corresponding Fitting ideal of each n -th Rees power E^n , and give an easy proof of the Burch's inequality for equimultiple modules.

In this paper we shall not use the notion of integral closure of modules. For the general aspects of this theory we refer to the corresponding chapters of the recent books by W. V. Vasconcelos [27] and I. Swanson and C. Huneke [25].

2. REDUCTION OF MODULES

In this section we review the notion of reduction of modules and state the results we shall use throughout this paper.

Suppose that E is a finitely generated torsionfree R -module having a rank over a Noetherian ring R . Let U be an R -submodule of E . U is said to be a *reduction* of E if

$$E^{r+1} = U \cdot E^r$$

for some $r \geq 0$ (this product taken inside $R(E)$). The least integer r for which $E^{r+1} = U \cdot E^r$ is called *the reduction number of E with respect to U* , and is denoted by $r_U(E)$. A reduction of E is called *minimal* if it is minimal with respect to inclusion.

It is clear that E is a reduction of itself with $r_E(E) = 0$. Moreover, if U is a reduction of E , then $U \otimes_R S$ is a reduction of $E \otimes_R S$ where S is any of the rings $R_{\mathfrak{p}}$ with \mathfrak{p} a prime ideal, $Q = \text{Quot}(R)$ or a polynomial ring. Further if U is a reduction and $E^{r+1} = U \cdot E^r$ for some $r \geq 0$ then $E^{n+1} = U \cdot E^n$ for all $n \geq r$.

Since $\mathcal{R}(E)$ is a standard graded algebra over R , one may also apply to this situation the notion of reduction for graded rings introduced by A. Ooishi in [21]. In fact, this is equivalent to the above notion of reduction of modules when the results in [21] are adequately read in our set up. Alternatively, it is possible to translate to the case of modules the results and proofs in [12, section 10] for ideals in order to obtain the basic properties of the theory of reduction of modules.

Recall that given a Noetherian local ring (R, \mathfrak{m}, k) the *fiber cone* of $\mathcal{R}(E)$ is the graded ring $\mathcal{F}(E) = \mathcal{R}(E)/\mathfrak{m}\mathcal{R}(E) = \bigoplus_{i \geq 0} E^i/\mathfrak{m}E^i$. The Krull dimension of $\mathcal{F}(E)$ is called the *analytic spread* of E and is denoted by $\ell(E)$. For an element $a \in E$ we denote by $\bar{a} = a + \mathfrak{m}E \in E/\mathfrak{m}E \subset \mathcal{F}(E)$. Then one can see that $U = Ra_1 + \dots + Ra_n$ is a reduction of E if and only if $\dim \mathcal{F}(E)/\langle \bar{a}_1, \dots, \bar{a}_n \rangle = 0$. In particular, we get $\mu(U) \geq \ell(E)$.

Next, we list the results on the theory of reduction of modules that we shall use later in this paper.

Proposition 2.1. *Let (R, \mathfrak{m}, k) be a Noetherian local ring, E a finitely generated torsionfree R -module having rank, U a reduction of E .*

- a) *There exists $V \subseteq U \subseteq E$ which is a minimal reduction of E .*
- b) *If $V \subseteq E$ is a minimal reduction of E and $V = \langle a_1, \dots, a_n \rangle$ with $n = \mu(V)$, then $\bar{a}_1, \dots, \bar{a}_n \in \mathcal{F}(E)$ are linearly independent, i.e. $\mathfrak{m}E \cap V = \mathfrak{m}V$.*
- c) *If $V \subseteq U \subseteq E$ is a minimal reduction of E and $V = \langle a_1, \dots, a_n \rangle$ with $n = \mu(V)$, then there exist $b_1, \dots, b_m \in E$ such that $\langle a_1, \dots, a_n, b_1, \dots, b_m \rangle = E$ and $\mu(E) = n + m$. In particular, $\mu(U) \geq \mu(V) \geq \ell(E)$.*
- d) *The following are equivalent:*
 - d1) *$V \subseteq E$ is a reduction and $\mu(V) = \ell(E)$.*
 - d2) *If $V = \langle a_1, \dots, a_n \rangle$ with $n = \mu(V)$, then $\bar{a}_1, \dots, \bar{a}_n \in \mathcal{F}(E)$ is a homogeneous system of parameters.*

And if any of these two equivalent conditions holds, V is a minimal reduction of E .

- e) *If the residue field k is infinite and $V \subseteq U$ is a minimal reduction, then conditions d1) and d2) hold. In particular, $\mathcal{F}(V) \subset \mathcal{F}(E)$ is a noether normalization of $\mathcal{F}(E)$ and $V^n \cap \mathfrak{m}E^n = \mathfrak{m}V^n$ for all $n \geq 0$.*

As a consequence, minimal reductions always exist. The *reduction number* of E , denoted by $r(E)$, is the minimum of $r_U(E)$, where U ranges over all minimal reductions of E .

Remark 2.2. If the residue field is finite, a minimal set of generators of a minimal reduction of E is not necessarily a homogeneous system of parameters of $\mathcal{F}(E)$. Nevertheless, there always exist homogeneous systems of parameters of $\mathcal{F}(E)$. This is equivalent to the existence of a family of elements $a_1 \in E^{r_1} \setminus \mathfrak{m}E^{r_1}, \dots, a_s \in E^{r_s} \setminus \mathfrak{m}E^{r_s}$, where $s = \ell(E)$, such that for some r , $E^r = a_1 E^{r-r_1} + \dots + a_s E^{r-r_s}$; and $\ell(E)$ is the minimum positive number for a such family of elements to exist.

Corollary 2.3. *Let (R, \mathfrak{m}, k) be a Noetherian local ring, E a finitely generated torsionfree R -module having rank. Then $\ell(E_{\mathfrak{p}}) \leq \ell(E)$ for all $\mathfrak{p} \in \text{Spec}(R)$*

Proof. Assume first that k is infinite. Let U be a minimal reduction of E and let $\mathfrak{p} \in \text{Spec}(R)$ be any prime. Then $U_{\mathfrak{p}}$ is a reduction of $E_{\mathfrak{p}}$ and so

$$\ell(E_{\mathfrak{p}}) \leq \mu(U_{\mathfrak{p}}) \leq \mu(U) = \ell(E).$$

Let R'' be a Nagata extension of R . Hence $\mathfrak{q} = \mathfrak{p}R'' \in \text{Spec}(R'')$ and $R''_{\mathfrak{q}}$ is a Nagata extension of $R_{\mathfrak{p}}$. Therefore, applying the above inequality

$$\ell(E_{\mathfrak{p}}) = \ell(E_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R''_{\mathfrak{q}}) = \ell(E''_{\mathfrak{q}}) \leq \ell(E'') = \ell(E).$$

□

A module E is said to be of *linear type* if $\mathcal{R}(E) = \mathcal{S}(E)$. Clearly, every finitely generated free module over a Noetherian ring is of linear type.

Next we observe that a module of linear type admits no proper reductions.

Corollary 2.4. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank. If E is of linear type then E has no proper reductions. In particular, if E is a free R -module then E has no proper reductions.*

Proof. Assume first that (R, \mathfrak{m}, k) is local. Then, we have

$$\ell(E) = \dim(\mathcal{R}(E) \otimes_R k) = \dim(\mathcal{S}(E) \otimes_R k) = \dim \mathcal{S}_k(E \otimes_R k).$$

Since $E \otimes_R k$ is a free k -module, $\text{rank}(E \otimes_R k) = \dim_k(E \otimes_R k) = \mu(E)$, and so $\ell(E) = \mu(E)$. By Proposition 2.1, E is a minimal reduction of itself. Hence $r(E) = 0$.

Let now $U \subseteq E$ be a reduction of E . Then, $U_{\mathfrak{m}} \subseteq E_{\mathfrak{m}}$ is a reduction of $E_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m} \subset R$, and by the local case, $U_{\mathfrak{m}} = E_{\mathfrak{m}}$. Therefore, $U = E$. □

Any reduction U of E has rank and $\text{rank } U = \text{rank } E$. Namely,

Proposition 2.5. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank and $U \subseteq E$ be a reduction of E .*

- a) U has rank and $\text{rank } U = \text{rank } E$;
- b) $\text{grade } E/U > 0$.

Proof. a) If $\text{rank } E = e$ and $Q = \text{Quot}(R)$, then $E \otimes_R Q \simeq Q^e$ and by Corollary 2.4 $E \otimes_R Q = U \otimes_R Q$, proving that $\text{rank } E = \text{rank } U$.

b) We have $(E/U)_{\mathfrak{p}} \simeq E_{\mathfrak{p}}/U_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \text{Ass } R$. But

$$(E/U)_{\mathfrak{p}} = 0 \Leftrightarrow \mathfrak{p} \notin \text{Supp } E/U = V(\text{ann}_R(E/U)) \Leftrightarrow \text{ann}_R(E/U) \not\subseteq \mathfrak{p}.$$

It follows that $\text{ann}_R(E/U) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Ass } R} \mathfrak{p} = \text{Z}(R)$, and so $\text{grade } E/U > 0$. □

We close this section mentioning the upper and lower bounds for the analytic spread obtained in [24] and deducing two easy consequences.

Proposition 2.6 ([24, Proposition 2.3]). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension $d > 0$ and E a finitely generated R -module having rank e . Then $e \leq \ell(E) \leq d + e - 1$.*

Corollary 2.7. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension $d > 0$ with infinite residue field and let E a finitely generated torsionfree R -module having rank e . Then $\ell(E) = e$ if and only if any minimal reduction of E is a free R -module.*

Proof. Let U be a minimal reduction of E . According to Proposition 2.1 and to Proposition 2.6

$$\mu(U) = \ell(E) \geq e = \text{rank } U.$$

Hence $\ell(E) = e$ if and only if $\text{rank } U = \mu(U)$ if and only if U is a free R -module. \square

Corollary 2.8. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension 1 with infinite residue field and let E a finitely generated torsionfree R -module having rank e . Then any minimal reduction of E is a free R -module.*

3. THE SUPPORT OF G/E AND IDEAL MODULES

Given a finitely generated torsionfree R -module E having positive rank e , E affords an embedding into a free module of the same rank, $E \subseteq G \simeq R^e$. The aim of this section is to realize the support of G/E as the variety of a certain ideal. To do this we first establish the inclusions

$$\text{Supp } E/U \subseteq V(F_e(E)) \subseteq \text{Supp } G/U,$$

where $F_e(E)$ is the e -Fitting ideal of E and U is a reduction of E . As we will see, equality in the second inclusion holds when $\text{grade } G/E \geq 2$, which determine a class of modules already introduced in [24] that afford a natural embedding into a free module of the same rank, the class of ideal modules. This type of modules behave similarly to an ideal, as we show.

Recall that $F_i(E) := I_{n-i}(\varphi)$ is the ideal generated by the $(n-i) \times (n-i)$ minors of φ where $R^m \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$ is a finite presentation of E , $0 \leq i \leq n$. If E has positive rank e then

$$F_0(E) = \cdots = F_{e-1}(E) = (0) \subsetneq F_e(E) \cdots \subseteq F_i(E) \subseteq \cdots \subseteq R.$$

Moreover, $\text{Supp } E = V(F_0(E))$ and, for every $\mathfrak{p} \in \text{Spec}(R)$, $\mu(E_{\mathfrak{p}}) = n$ if and only if $F_{n-1}(E) \subseteq \mathfrak{p}$ and $F_n(E) \not\subseteq \mathfrak{p}$.

Using these properties and the fact that, when (R, \mathfrak{m}) is local, a finitely generated module E with rank $e > 0$ is free if and only if $\mu(E) = e$, one immediately gets that if R is a Noetherian ring and E a finitely generated R -module having rank $e > 0$, then the free locus of E is given by $\text{Spec}(R) \setminus V(F_e(E))$, and coincides with the locus of prime ideals $\mathfrak{p} \in \text{Spec}(R)$ such that $\mu(E_{\mathfrak{p}}) \leq e$. In particular, if (R, \mathfrak{m}) is also a local ring, then E is free if and only if $F_e(E) = R$.

Applying the above facts we observe the following about grade $F_e(E)$. For its proof just use that $\text{grade } F_e(E) = \inf\{\text{depth } R_{\mathfrak{p}} \mid \mathfrak{p} \in V(F_e(E))\} > 0$.

Lemma 3.1. *Let E be a finitely generated module over a Noetherian ring R having positive rank e . Then:*

- a) $\text{grade } F_e(E) > 0$.
- b) $\text{grade } F_e(E) \geq 2$ if and only if $\mu(E_{\mathfrak{p}}) \leq e$ for every $\mathfrak{p} \in \text{Spec}(R)$ whenever $\text{depth } R_{\mathfrak{p}} = 1$.
- c) If $\text{grade } F_e(E) \geq 2$ then $E_{\mathfrak{p}}$ is free whenever $\text{depth } R_{\mathfrak{p}} \leq 1$.

Next we prove that $\text{grade } E/U \geq \text{grade } F_e(E)$ for any reduction U of E , $e = \text{rank } E > 0$.

Proposition 3.2. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank $e > 0$ and U a reduction of E . Then $\text{Supp } E/U \subseteq V(F_e(E))$.*

Proof. Let $\mathfrak{p} \notin V(F_e(E))$. Hence $E_{\mathfrak{p}}$ is free and, since $U_{\mathfrak{p}}$ is a reduction of $E_{\mathfrak{p}}$, we must have $U_{\mathfrak{p}} = E_{\mathfrak{p}}$ (because a free module has no proper reductions). Therefore $\mathfrak{p} \notin \text{Supp } E/U$. It follows that $\text{Supp } E/U \subseteq V(F_e(E))$. \square

Let E be a finitely generated torsionfree R -module having rank $e > 0$. Hence E is a submodule of a free R -module $G \simeq R^e$, and so for any reduction U of E , $U \subseteq E \subseteq G$. In this case, we show that $V(F_e(E)) \subseteq \text{Supp } G/U = \text{Supp } G/E$ with equality if $\text{grade } G/U \geq 2$.

Proposition 3.3. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank $e > 0$ and U a reduction of E . Suppose that $E \subseteq G \simeq R^e$. Then*

- a) $\text{Supp } E/U \subseteq V(F_e(E)) \subseteq \text{Supp } G/U = \text{Supp } G/E$;
- b) $\text{grade } E/U \geq \text{grade } F_e(E) \geq \text{grade } G/U = \text{grade } G/E$.

Proof. Let U be a reduction of E . We first note that if $U = G$ then $U = E = G$, and so $G/U = 0$ and $F_e(E) = R$. In this case, the formula reads $V(R) = \text{Supp}(0) = \emptyset$. Hence, we may assume that $U \neq G$. Let $\mathfrak{p} \in \text{Spec}(R)$ be arbitrary. Since

$$\begin{aligned} \mathfrak{p} \notin \text{Supp } G/U &\iff G_{\mathfrak{p}}/U_{\mathfrak{p}} = (G/U)_{\mathfrak{p}} = 0 \iff U_{\mathfrak{p}} = E_{\mathfrak{p}} = G_{\mathfrak{p}} \\ &\implies E_{\mathfrak{p}} \text{ is free} \iff F_e(E) \not\subseteq \mathfrak{p}, \end{aligned}$$

then $V(F_e(E)) \subseteq \text{Supp } G/U$. We also get $V(F_e(E)) \subseteq \text{Supp } G/E$, hence

$$\text{Supp } E/U \subseteq V(F_e(E)) \subseteq \text{Supp } G/E.$$

Now from the exact sequence $0 \rightarrow E/U \rightarrow G/U \rightarrow G/E \rightarrow 0$ we have

$$\text{Supp } G/U = \text{Supp } E/U \cup \text{Supp } G/E = \text{Supp } G/E,$$

and a) is proved. b) follows as a direct consequence of a). \square

In the situation of Proposition 3.3, we observe that if $E_{\mathfrak{p}}$ is free then $E_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^e \simeq G_{\mathfrak{p}}$; however, in general, we may have $E_{\mathfrak{p}} \subsetneq G_{\mathfrak{p}}$. Next we give a sufficient condition to guarantee the equality.

Lemma 3.4. *Let R be a Noetherian ring, $F \subseteq G$ finitely generated free R -modules having the same rank. If $\text{grade } G/F \geq 2$ then $F = G$.*

Proof. Suppose that $F \subsetneq G$. Hence $\text{Ass } G/F \neq \emptyset$. Let $\mathfrak{p} \in \text{Ass } G/F$. Since $\text{grade } G/F \geq 2$ then $\text{depth } R_{\mathfrak{p}} \geq 2$. Supposing $\text{rank } F = e = \text{rank } G$ then $F_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^e \simeq G_{\mathfrak{p}}$, and so

$$\text{depth } F_{\mathfrak{p}} = \text{depth } G_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} \geq 2.$$

It follows that

$$\text{depth } G_{\mathfrak{p}}/F_{\mathfrak{p}} \geq \min\{\text{depth } G_{\mathfrak{p}}, \text{depth } F_{\mathfrak{p}} - 1\} \geq 1,$$

contradicting $\text{depth } G_{\mathfrak{p}}/F_{\mathfrak{p}} = 0$. Hence $\text{Ass } G/F = \emptyset$, and so $F = G$. \square

Proposition 3.5. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank $e > 0$ and U a reduction of E . Suppose that $E \subseteq G \simeq R^e$. If $\text{grade } G/U \geq 2$ then*

- a) $V(F_e(E)) = \text{Supp } G/U$;
- b) $\text{grade } G/U = \text{grade } F_e(E)$.

Proof. a) By Proposition 3.3, $V(F_e(E)) \subseteq \text{Supp } G/U$. For the other inclusion let $\mathfrak{p} \in \text{Spec}(R) \setminus V(F_e(E))$. Hence $E_{\mathfrak{p}}$ is free, and so $U_{\mathfrak{p}} = E_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^e \simeq G_{\mathfrak{p}}$. By assumption

$$\text{grade } G_{\mathfrak{p}}/U_{\mathfrak{p}} = \text{grade } (G/U)_{\mathfrak{p}} \geq \text{grade } G/U \geq 2.$$

Hence, by the previous lemma, $U_{\mathfrak{p}} = G_{\mathfrak{p}}$ and so $\mathfrak{p} \notin \text{Supp } G/U$. The equality holds. b) follows by a). \square

Theorem 3.6. *Let R be a Noetherian ring, E a finitely generated torsionfree R -module having rank $e > 0$. Suppose that $E \subseteq G \simeq R^e$. The following are equivalent:*

- a) $\text{grade } G/E \geq 2$;
- b) $\text{grade } G/U \geq 2$ for any reduction U of E ;
- c) $\text{grade } G/U \geq 2$ for some reduction U of E .

If any of the above conditions holds, $V(F_e(E)) = V(F_e(U)) = \text{Supp } G/E = \text{Supp } G/U$ for any reduction U of E ; in particular, $\text{grade } G/E = \text{grade } F_e(E) = \text{grade } F_e(U) = \text{grade } G/U$.

Proof. This follows by Proposition 3.3 and by Proposition 3.5 (applied twice). \square

In general, the class of modules of the form $E \subseteq G \simeq R^e$ with $\text{grade } G/E \geq 2$ is sufficiently special to have a name: *ideal module*. This definition of ideal module is one of the various characterizations of ideal module in [24, Proposition 5.1-c)] of Simis-Ulrich-Vasconcelos, where ideal modules are defined as the finitely generated and torsion free R -modules E , such that the double dual E^{**} is free. These type of modules behave similarly to an ideal, because they afford a natural embedding into a free module of the same rank, its bidual. See [24] for details.

It is worthwhile to point out that although the definition of ideal module is intrinsic and does not depend on the possible embedding of E into a free module G , the property $\text{grade } G/E \geq 2$ depends on the chosen embedding as the following simple example shows: Let $R = k[[x, y, z]]$ where k is a field and $E = (zx, zy)$. Then $E \simeq I = (x, y)$ (as R -modules) and so $E^{**} \simeq I^{**}$ which is free because I is an ideal of grade 2. Thus E is an ideal module. On the other hand, $\text{grade } R/E = 1$ because E is an ideal of grade 1. In this case, the "right" embedding for E is given by $E \simeq I = (x, y) \subset G := R$.

It is also clear that any reduction U of an ideal module E having rank e is an ideal module having rank e . Moreover, by Proposition 3.3, $\text{grade } E/U \geq 2$.

Ideal modules satisfy the following properties which are easy to prove.

Remark 3.7. Let R be a Noetherian ring, $\mathfrak{p} \in \text{Spec}(R)$, E an ideal module, and G a finitely generated free module containing E with $\text{grade } G/E \geq 2$. Then

- a) E has rank and $\text{rank } E = \text{rank } G$;

- b) $\text{grade } F_e(E) \geq 2$, where $e = \text{rank } E$;
- c) $E_{\mathfrak{p}}$ is an ideal module;
- d) $E_{\mathfrak{p}}$ is free whenever $\text{depth } R_{\mathfrak{p}} \leq 1$.

For any reduction U of an ideal module E having rank e contained in a free module $G \simeq R^e$ we have natural isomorphisms $U^{**} \simeq E^{**} \simeq G^{**} \simeq G$. To see that one just need to apply the following lemma:

Lemma 3.8. *Let R be a Noetherian ring and $E_2 \subseteq E_1$ finitely generated R -modules such that $\text{grade } E_1/E_2 \geq 2$. Then, $E_2^{**} \simeq E_1^{**}$.*

Proof. Dualizing the exact sequence $0 \rightarrow E_2 \hookrightarrow E_1 \twoheadrightarrow E_1/E_2 \rightarrow 0$ we obtain the exact sequence

$$0 \rightarrow (E_1/E_2)^* \rightarrow E_1^* \rightarrow E_2^* \rightarrow \text{Ext}_R^1(E_1/E_2, R).$$

Since $\text{grade } E_1/E_2 \geq 2$, $(E_1/E_2)^* = \text{Ext}_R^1(E_1/E_2, R) = 0$, and so $E_1^* \simeq E_2^*$. Therefore $E_2^{**} \simeq E_1^{**}$. \square

Proposition 3.9. *Let R be a Noetherian ring and E an ideal module of rank e . Then all free R -modules $R^e \simeq G_i \supseteq E$ with $\text{grade } G_i/E \geq 2$ are incomparable for inclusion and $G_i \simeq E^{**}$.*

Proof. Suppose that $E \subset G_i \subseteq G_j$ with $G_i, G_j \simeq R^e$, $\text{grade } G_i/E \geq 2$, $\text{grade } G_j/E \geq 2$. Hence $\text{grade } G_j/G_i \geq 2$, and so $G_i = G_j$ by Lemma 3.4. The last assertion follows by Lemma 3.8. \square

Next we observe that over a Noetherian local ring (R, \mathfrak{m}) with $\text{depth } R \geq 2$, if $\dim G/E = 0$ then $\text{grade } G/E \geq 2$ and E is an ideal module. In particular, any \mathfrak{m} -primary R -ideal I will be an ideal module.

Proposition 3.10. *Let R be a Noetherian local ring with $\text{depth } R \geq 2$ and let $E \subsetneq G \simeq R^e$ be an R -module. If $\dim G/E = 0$ then $\text{grade } G/E \geq 2$. In particular E is an ideal module.*

Proof. By assumption $\dim G/E = 0$ and $\text{depth } R \geq 2$. Hence

$$\text{Hom}_R(G/E, R) = 0 = \text{Ext}_R^1(G/E, R),$$

by [19, Theorem 17.1]. Thus $\text{grade } G/E \geq 2$, and so E is an ideal module. \square

In the following, we determine $\dim G/E$ and $\text{depth } G/E$. In particular, we observe that any ideal module has maximum Krull dimension.

Proposition 3.11. *Let R be a Noetherian local ring, $\dim R = d \geq 2$, E an ideal module over R and U a reduction of E . Suppose that $E \subsetneq G \simeq R^e$, $e > 0$. Then*

- a) $\dim G/E \leq d - \text{ht } F_e(E) \leq d - 2$;
- b) $\dim E = d$;
- c) if in addition R is Cohen-Macaulay, then

$$\text{depth } E - 1 = \text{depth } G/E \leq \dim G/E = d - \text{ht } F_e(E).$$

Proof. a) We have $\text{Supp } G/E = V(F_e(E)) = \text{Supp } R/F_e(E)$ and so

$$\dim G/E = \dim R/F_e(E) \leq d - \text{ht } F_e(E) \leq d - 2.$$

b) Since G is free and $\dim G/E < d$, then

$$d = \dim R = \dim G = \max\{\dim E, \dim G/E\} = \dim E,$$

as required.

c) For the first equality we apply the depth Lemma to the exact sequence $0 \rightarrow E \rightarrow G \rightarrow G/E \rightarrow 0$. Now, c) follows by a). \square

As stated in Proposition 2.6, the analytic spread of a finitely generated module E having rank e , over a d -dimensional Noetherian local ring, satisfies the inequalities

$$e \leq \ell(E) \leq d + e - 1.$$

Now we deduce another lower bound for the analytic spread, for any torsionfree module, and as a consequence we recover the one stated in [24, Proposition 5.2] in the case where E is, in addition, an ideal module.

Proposition 3.12. *Let R be a Noetherian local ring and $E \subseteq G \simeq R^e$ a finitely generated torsionfree R -module having rank $e > 0$, but not free. Then*

$$\ell(E) \geq \text{ht } F_0(G/U) + e - 1,$$

for any minimal reduction U of E . In particular, if $\text{grade } G/E \geq 2$ then $\ell(E) \geq \text{ht } F_e(E) + e - 1 \geq e + 1$.

Proof. We may assume that the residue field of R is infinite, since any Nagata extension R'' of R has infinite residue field and, for any finitely generated R -module M , $\ell(M \otimes_R R'') = \ell(M)$, $\text{rank}(M \otimes_R R'') = \text{rank } M$ and $\text{ht } F_i(M \otimes_R R'') = \text{ht } F_i(M)R'' = \text{ht } F_i(M)$.

Let U be a minimal reduction of E and suppose that $\mu(U) = n$ (hence $\ell(E) = n$). Then there exists an R -epimorphism $\psi: R^n \rightarrow U$. Further, since E is not free, U is a (proper) submodule of G . Therefore, we have an exact sequence

$$(1) \quad R^n \xrightarrow{\psi} G \rightarrow G/U \rightarrow 0.$$

By the Eagon-Northcott Theorem (see [26, Theorem 1.1.16]),

$$\text{ht } F_0(G/U) = \text{ht } I_e(\psi) \leq n - e + 1 = \ell(E) - e + 1,$$

proving the inequality. Moreover, if $\text{grade } G/E \geq 2$ then $\text{ht } F_0(G/U) = \text{ht } F_e(E) \geq 2$, (by Theorem 3.6), and the other inequalities follow. \square

If (R, \mathfrak{m}) is a Noetherian local ring and I an \mathfrak{m} -primary ideal then the analytic spread is the biggest possible: $\ell(I) = \dim R$. Let E be a finitely generated R -module having rank $e > 0$, but not free. Since the free locus of E is given by $\text{Spec}(R) \setminus V(F_e(E))$ we have that E is free locally on the punctured spectrum, that is $E_{\mathfrak{p}}$ is free for every prime $\mathfrak{p} \neq \mathfrak{m}$, if and only if $F_e(E)$ is an \mathfrak{m} -primary ideal. As a consequence, we get the formula given in [24, Proposition 5.2] for ideal modules which are free locally on the punctured spectrum.

Corollary 3.13. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d and let E be an ideal module having rank $e > 0$ which is free locally on the punctured spectrum. Then $\ell(E) = d + e - 1 = \text{ht } F_e(E) + e - 1$.*

We note here that to be free locally on the punctured spectrum is not a sufficient condition for a module to be an ideal module, as the following simple example shows: Let $R = k[[x, y, z]]$ where k is a field and $E = Re_1 \oplus Re_2 \oplus Re_3 / (xe_1 + ye_2 + ze_3)$, where e_1, e_2, e_3 is the canonical basis of R^3 . Then, $\text{rank } E = 2$, $F_2(E) = (x, y, z)$ and E has projective dimension 1. Thus E is free locally on the punctured spectrum. On the other hand, by [3, Proposition 1.4.1] E is reflexive and so E^{**} is not free.

In the case where R is a Noetherian local ring with depth $R \geq 2$ we proved that if $\dim G/E = 0$ then E is an ideal module (cf. Proposition 3.10). Thus we have the following equivalence.

Proposition 3.14. *Let (R, \mathfrak{m}) be a Noetherian local ring with depth $R \geq 2$ and let $E \subsetneq G \simeq R^e$ be an R -module having rank $e > 0$. Then E is free locally on the punctured spectrum with grade $G/E \geq 2$ if and only if $\dim G/E = 0$.*

Proof. Suppose that $E \subsetneq G \simeq R^e$ is free locally on the punctured spectrum and that $\text{grade } G/E \geq 2$. Hence $E_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^e \simeq G_{\mathfrak{p}}$ for each prime $\mathfrak{p} \neq \mathfrak{m}$. Since $\text{grade } G_{\mathfrak{p}}/E_{\mathfrak{p}} \geq \text{grade } G/E \geq 2$ then $E_{\mathfrak{p}} = G_{\mathfrak{p}}$ (by Lemma 3.4). Therefore $\text{Supp } G/E = \text{Ass } G/E = \{\mathfrak{m}\}$, and so $\dim G/E = 0$. The converse follows by Proposition 3.10. \square

Corollary 3.15. *Let (R, \mathfrak{m}) be a Noetherian local ring with depth $R \geq 2$ and let E be an ideal module having rank $e > 0$ which is free locally on the punctured spectrum. Then any reduction U of E is free locally on the punctured spectrum.*

Proof. Suppose that $E \subsetneq G \simeq R^e$ with $\text{grade } G/E \geq 2$. Let U be a reduction of E . Hence

$$\text{Supp } G/U = V(F_e(E)) = \text{Supp } G/E = \{\mathfrak{m}\}$$

(by Theorem 3.5, Proposition 3.11). Therefore $\dim G/U = 0$ and U is free locally on the punctured spectrum. \square

In the case of dimension 2 every ideal module is free locally on the punctured spectrum.

Corollary 3.16. *Let R be a Cohen-Macaulay local ring, $\dim R = 2$, $E \subsetneq G \simeq R^e$, $e > 0$ an ideal module over R . Then E is free locally on the punctured spectrum.*

Proof. By Proposition 3.11, $\dim G/E = 0$, and the assertion follows by Proposition 3.14. \square

4. DEVIATION AND ANALYTIC DEVIATION

In this section we define the deviation and the analytic deviation for a module. These invariants give rise to the notions of complete intersection, equimultiple, and generically a complete intersection for modules, as in the case of ideals.

Let (R, \mathfrak{m}) be a Noetherian local ring and I an R -ideal. Recall that the *deviation* of I is defined to be the difference $d(I) = \mu(I) - \text{ht } I$, whereas the *analytic deviation* of I is the difference $\text{ad}(I) = \ell(I) - \text{ht } I$. We always have $d(I) \geq 0$ (by Krull's Principal Ideal Theorem) and $\text{ad}(I) \geq 0$. As a matter of fact, we have

$$\mu(I) \geq \ell(I) \geq \text{ht } I$$

(cf. [12, Proposition 10.20]). In the case where $d(I) = 0$, I is called a *complete intersection* and if $\text{ad}(I) = 0$, I is said to be *equimultiple*. Furthermore, if $\mu(I_{\mathfrak{p}}) = \text{ht } I$ for all minimal prime ideals $\mathfrak{p} \in \text{Min } R/I$, I is called *generically a complete intersection*.

For non-free modules we have the following definitions.

Definition. Let R be a Noetherian local ring and E a finitely generated R -module having rank $e > 0$ but not free. We define the *deviation* of E by $d(E) = \mu(E) - e + 1 - \text{ht } F_e(E)$ and the *analytic deviation* of E by $\text{ad}(E) = \ell(E) - e + 1 - \text{ht } F_e(E)$.

We notice that our definitions slightly differ from those in [24], since we use $\text{ht } F_e(E)$ instead of $\text{grade } F_e(E)$. Clearly, they coincide in the Cohen-Macaulay case.

Applying Proposition 2.1 and Proposition 3.12 we get the following.

Remark 4.1. Let R be a Noetherian local ring and E an ideal module having rank e , but not free. Then $d(E) \geq \text{ad}(E) \geq 0$.

In accordance with the previous remark we have the following definitions for non-free ideal modules.

Definition. Let R be a Noetherian local ring and E a non-free ideal module over R of rank e . We say that E is:

1. a *complete intersection module* if $d(E) = 0$,
2. an *equimultiple module* if $\text{ad}(E) = 0$,
3. *generically a complete intersection module* if $\mu(E_{\mathfrak{p}}) = \text{ht } F_e(E) + e - 1$ for all minimal prime ideals $\mathfrak{p} \in \text{Min } R/F_e(E)$.

As expected, as in the case of \mathfrak{m} -primary ideals we have the following example of equimultiple modules.

Example 4.2. Let R be a Noetherian local ring with $\dim R = d > 0$ and E be a non-free ideal module which is free locally on the punctured spectrum. Then, by Corollary 3.13 E is equimultiple.

Complete intersection modules were already considered by D. Buchsbaum and D. Rim [4] and by D. Katz and C. Naude [14], in particular situations. In fact, Katz-Naude defined a *module of principal class* $E \subseteq R^e$ if it is generated by $n \geq e$ column vectors and $\text{ht } F_e(E) = n - e + 1$. If, in addition, R is a local ring and E is embedded into a free module G such that the quotient G/E has finite length, then the $e \times n$ matrix whose columns correspond to the generators of E is a parameter matrix in the sense of [4], and E is called a *parameter module*.

Clearly, if R is a local ring, an ideal module is of principal class if and only if it is a complete intersection. Moreover, in virtue of Proposition 3.14, if $\text{depth } R \geq 2$ any non-free parameter module having positive rank is a complete intersection and also free locally on the punctured spectrum.

As in the case of ideals we have the following relations, that we list here for completeness.

Proposition 4.3. *Let (R, \mathfrak{m}, k) be a Noetherian local ring with $\dim R = d > 0$ and E a non-free ideal module having rank e .*

- a) *If E is a complete intersection then:*
 - i) *E is equimultiple;*
 - ii) *E is generically a complete intersection;*
 - iii) *$\text{ht } F_e(E_{\mathfrak{p}}) = \text{ht } F_e(E)$, $\mu(E_{\mathfrak{p}}) = \mu(E)$ for every $\mathfrak{p} \in \text{Spec}(R)$;*
 - iv) *$E_{\mathfrak{p}}$ is a complete intersection for every $\mathfrak{p} \in \text{Spec}(R)$.*
- b) *E is generically a complete intersection if and only if*
 - i) *$E_{\mathfrak{p}}$ is a complete intersection for every $\mathfrak{p} \in \text{Min } R/F_e(E)$, and*
 - ii) *$\text{ht } F_e(E_{\mathfrak{p}}) = \text{ht } F_e(E)$ for every $\mathfrak{p} \in \text{Min } R/F_e(E)$;*
- c) *If there exists a reduction U of E which is a complete intersection then E is equimultiple.*
- d) *If k is infinite, then E is equimultiple if and only if every minimal reduction U of E is a complete intersection.*
- e) *If E is a complete intersection then E is equimultiple, E admits no proper reductions and $r(E) = 0$. If k is infinite also the converse holds.*
- f) *Suppose that E is free locally on the punctured spectrum. Then E is a complete intersection if and only if $\mu(E) = d + e - 1$.*

Proof. a) The first assertion is immediate by Remark 4.1. For the others, let $\mathfrak{p} \in \text{Spec}(R)$ be arbitrary. Then we have,

$$\mu(E_{\mathfrak{p}}) - e + 1 \geq \text{ht } F_e(E_{\mathfrak{p}}) \geq \text{ht } F_e(E) = \mu(E) - e + 1 \geq \mu(E_{\mathfrak{p}}) - e + 1,$$

and so (ii)-(iv) hold.

b) Suppose that E is generically a complete intersection module. Let $\mathfrak{p} \in \text{Min } R/F_e(E)$. Hence, $E_{\mathfrak{p}}$ is an ideal module over $R_{\mathfrak{p}}$ having rank e , and we have

$$\text{ht } F_e(E) \leq \text{ht } F_e(E_{\mathfrak{p}}) \leq \ell(E_{\mathfrak{p}}) - e + 1 \leq \mu(E_{\mathfrak{p}}) - e + 1 = \text{ht } F_e(E).$$

Therefore, $\text{ht } F_e(E) = \text{ht } F_e(E_{\mathfrak{p}}) = \mu(E_{\mathfrak{p}}) - e + 1$ and $E_{\mathfrak{p}}$ is a complete intersection. The converse is clear.

c) Let U be a reduction of E which is a complete intersection module. Since E is an ideal module we have, by Theorem 3.6 and by Proposition 2.1

$$\text{ht } F_e(E) = \text{ht } F_e(U) = \mu(U) - e + 1 \geq \ell(E) - e + 1 \geq \text{ht } F_e(E),$$

proving that E is equimultiple.

d) follows by Theorem 3.6 and by Proposition 2.1.

e) Suppose that E is a complete intersection. By a) E is equimultiple. Moreover,

$$\text{ht } F_e(E) \leq \ell(E) - e + 1 \leq \mu(E) - e + 1 = \text{ht } F_e(E),$$

and so $\mu(E) = \ell(E)$. Hence E is a minimal reduction of itself (cf. Proposition 2.1). For the converse we have, by assumptions, $\mu(E) = \ell(E) = \text{ht } F_e(E) + e - 1$, proving that E is a complete intersection.

f) follows by Corollary 3.13. \square

We may construct complete intersection [resp. equimultiple or generically a complete intersection] modules of any rank $e \geq 2$ using ideals of the same type. First we note that if $E \simeq F \oplus E'$ is a finitely generated torsionfree R -module having rank, where F is a free R -module of rank e , then $\mathcal{R}(E) \simeq \mathcal{R}(E')[t_1, \dots, t_e]$.

Corollary 4.4. *Let (R, \mathfrak{m}) be a Noetherian local ring with $\dim R = d > 0$. Suppose that $E = F \oplus I$ where $F \simeq R^{e-1}$ and I an R -ideal with $\text{grade } I \geq 2$. Then E is a complete intersection [resp. equimultiple or generically a complete intersection] module if and only if I is a complete intersection [resp. equimultiple or generically a complete intersection] ideal.*

Proof. We have $V(F_e(E)) = V(F_1(I)) = V(I)$, hence $\text{ht } F_e(E) = \text{ht } I$. Moreover, $\mu(E) = \mu(I) + e - 1$ and $\ell(E) = \dim \mathcal{F}(E) = \dim \mathcal{F}(I)[t_1, \dots, t_{e-1}] = \ell(I) + e - 1$. It follows that E is a complete intersection [equimultiple] module if and only if I is a complete intersection [equimultiple] ideal. For generically a complete intersection modules apply Proposition 4.3. \square

5. EQUIMULTIPLE VERSUS COMPLETE INTERSECTION

Complete intersection modules have good properties. In fact, Simis-Ulrich-Vasconcelos showed that, in this case, $\mathcal{R}(E)$ is Cohen-Macaulay ([24, Corollary 5.6]) and Katz-Naude had proved that G/E is a perfect module ([14, Proposition 3.3]). Hence if R is Cohen-Macaulay and E a complete intersection R -module such that that $E \subsetneq G \simeq R^e$, then G/E is Cohen-Macaulay and $\text{grade } G/E = \text{proj dim } G/E = \text{ht } F_e(E)$.

We establish now few additional properties and prove several criteria for an equimultiple module being a complete intersection.

We observed that $\ell(E) \geq \text{ht } F_0(G/U) + e - 1$ for any non-free R -module $E \subsetneq G \simeq R^e$ of rank $e > 0$ and any minimal reduction U (cf. Proposition 3.12). In the case where E is equimultiple and $\text{grade } G/E \geq 2$ it is clear that the equality holds. Moreover, in this case, if R a Cohen-Macaulay local ring, we show that $F_0(G/U)$ is a perfect ideal and all the associated primes of $F_0(G/U)$ have the same height which is equal to $\ell(E) - e + 1 = \text{ht } F_e(E)$.

Proposition 5.1. *Let R be a Cohen-Macaulay local ring, $\dim R = d > 0$ and E a complete intersection module having rank $e > 0$. Suppose that $E \subsetneq G \simeq R^e$. Then*

- a) $F_0(G/E)$ is a perfect ideal;
- b) $\text{depth } G/E = \text{depth } R/F_0(G/E)$;
- c) $\text{Ass } G/E = \{\mathfrak{p} \in V(F_e(E)) \mid \text{ht } \mathfrak{p} = \ell(E) - e + 1\} = \text{Ass } R/F_0(G/E) = \text{Min } R/F_0(G/E) = \text{Min } R/F_e(E)$.

Proof. a), b) We may assume that the residue field of R is infinite. Suppose that $n = \mu(E) (= \ell(E))$. Since E is complete intersection (hence ideal module) and R is Cohen-Macaulay,

$$n - e + 1 = \text{ht } F_e(E) = \text{ht } F_0(G/E) = \text{grade } F_0(G/E)$$

-the second equality by Proposition 3.5. Hence, by [2, Theorem 2.7], $F_0(G/E) = I_e(\psi)$, with ψ as in (1), is a perfect ideal. Therefore, by the Auslander-Buchsbaum formula and since R is Cohen-Macaulay,

$$\begin{aligned} d - \text{depth } G/E &= \ell(E) - e + 1 = \text{proj dim } R/F_0(G/E) \\ &= d - \text{depth } R/F_0(G/E) \end{aligned}$$

and b) follows.

c) By Proposition 3.5, $\text{Min } R/F_0(G/E) = \text{Min } R/F_e(E)$. On the other hand, $R/F_0(G/E)$ is a Cohen-Macaulay local ring (by [3, Theorem 2.1.5]). Thus $\text{Min } R/F_0(G/E) = \text{Ass } R/F_0(G/E)$. Since E is a complete intersection, $\text{ht } F_e(E) = \text{ht } F_e(E_{\mathfrak{p}})$ and $E_{\mathfrak{p}}$ is a complete intersection, for every prime $\mathfrak{p} \in \text{Spec}(R)$ (by Proposition 4.3). Hence, $G_{\mathfrak{p}}/E_{\mathfrak{p}}$ is Cohen-Macaulay (by the previous result). Therefore $\text{depth } G_{\mathfrak{p}}/E_{\mathfrak{p}} = \text{ht } \mathfrak{p} - \text{ht } F_e(E_{\mathfrak{p}}) = \text{ht } \mathfrak{p} - \text{ht } F_e(E)$. Moreover, by b), $\text{Ass } R/F_0(G/E) = \text{Ass } R/F_e(E)$, and the equalities follow. \square

Since any minimal reduction of an equimultiple module is a complete intersection, we may assert the following.

Corollary 5.2. *Let R be a Cohen-Macaulay local ring, $\dim R = d > 0$ and E a equimultiple module. Suppose that $E \subsetneq G \simeq R^e$. Then, for every minimal reduction U of E ,*

$$\text{Ass } E/U \subseteq \text{Min } R/F_e(E) = \{\mathfrak{p} \in V(F_e(E)) \mid \text{ht } \mathfrak{p} = \ell(E) - e + 1\}.$$

Proof. We may assume that R has infinite residue field.

Let U be a minimal reduction of E . Since U is complete intersection,

$$\text{Ass } E/U \subseteq \text{Ass } G/U = \text{Min } R/F_e(U) = \text{Min } R/F_e(E).$$

(by Proposition 5.1). Moreover,

$$\begin{aligned} \mathfrak{p} \in \text{Min } R/F_e(E) &\iff \mathfrak{p} \in \text{Ass } G/U \iff \text{depth } G_{\mathfrak{p}}/U_{\mathfrak{p}} = 0 \\ &\iff \dim G_{\mathfrak{p}}/U_{\mathfrak{p}} = 0 \iff \text{ht } \mathfrak{p} = \text{ht } F_e(U_{\mathfrak{p}}). \end{aligned}$$

(by Proposition 3.11). Therefore,

$$\ell(E) - e + 1 = \text{ht } F_e(E) \leq \text{ht } \mathfrak{p} \leq \mu(U_{\mathfrak{p}}) - e + 1 \leq \mu(U) - e + 1 = \ell(E) - e + 1.$$

It follows that $\text{Min } R/F_e(E)$ is the set of all prime ideals $\mathfrak{p} \in V(F_e(E))$ such that $\text{ht } \mathfrak{p} = \ell(E) - e + 1$. \square

The following result extends to ideal modules a known criterion in the ideal case by of D. Eisenbud, M. Hermann and W. Vogel (see [8, Theorem p. 179]).

Theorem 5.3. *Let R be a Cohen-Macaulay local ring, E a non-free finitely generated torsionfree R -module having rank $e > 0$. Suppose that E is generically a complete intersection. Then E is complete intersection if and only if E is equimultiple.*

Proof. Since E is complete intersection [resp. equimultiple] if and only if $E'' = E \otimes_R R''$ is complete intersection [resp. equimultiple] for any Nagata extension R'' of R , we may assume that R has infinite residue field.

It is enough to prove that E equimultiple implies that E is complete intersection. So assume that E is equimultiple and let U be a minimal reduction of E . Hence U is a complete intersection. Suppose that $U \subsetneq E$. Let $\mathfrak{p} \in \text{Min } R/F_e(E)$. Hence $U_{\mathfrak{p}}$ is a reduction of $E_{\mathfrak{p}}$ and, since E is generically a complete intersection, $E_{\mathfrak{p}} = U_{\mathfrak{p}}$. In particular $E_{\mathfrak{p}} = U_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass } E/U$ (by Corollary 5.2) - a contradiction. Therefore $E = U$ (and $\text{Ass } E/U = \emptyset$). \square

Corollary 5.4. *Let R be a Cohen-Macaulay local ring, $\dim R = d > 0$ and $E \subsetneq G \simeq R^e$ an ideal module. Assume that $\text{ht } \mathfrak{p} = \ell(E) - e + 1$ for every $\mathfrak{p} \in \text{Min } R/F_e(E)$ and that $E_{\mathfrak{p}}$ is a complete intersection for every prime $\mathfrak{p} \in \text{Min } R/F_e(E)$. Then E is a complete intersection.*

Proof. Since $\text{ht } F_e(E) = \text{ht } \mathfrak{p}$ for some $\mathfrak{p} \in \text{Min } R/F_e(E)$ then $\ell(E) = \text{ht } F_e(E) + e - 1$ and E is equimultiple. Moreover, for any $\mathfrak{p} \in \text{Min } R/F_e(E)$

$$\ell(E_{\mathfrak{p}}) = \mu(E_{\mathfrak{p}}) = \text{ht } F_e(E_{\mathfrak{p}}) + e - 1 \geq \text{ht } F_e(E) + e - 1 = \ell(E) \geq \ell(E_{\mathfrak{p}}),$$

and so $\text{ht } F_e(E) = \text{ht } F_e(E_{\mathfrak{p}})$, proving that E is generically a complete intersection. Therefore, by Theorem 5.3, E is a complete intersection. \square

In [20, Théorème 2]), A. Micali proved that (R, \mathfrak{m}) is regular local if and only if $\mathcal{S}(\mathfrak{m})$ is a domain. This result was an important motivation to study the linear type property. We now prove a criterion for an equimultiple module to be a complete intersection that extends the above result of Micali.

Proposition 5.5. *Let R be a Noetherian local ring and let E be an ideal module. Then*

- a) *E is a complete intersection if and only if E is equimultiple and of linear type.*
- b) *If $\mathcal{S}(E)$ is a domain then E is a complete intersection if and only if E is equimultiple.*

Proof. We may assume that $k = R/\mathfrak{m}$ is infinite because any Nagata extension R'' of R has infinite residue field, $\mathcal{S}(E'') \simeq \mathcal{S}(E) \otimes_R R''$ and $\mathcal{R}(E'') \simeq \mathcal{R}(E) \otimes_R R''$. Suppose that E is equimultiple. Then every minimal reduction of E is a complete intersection (by Proposition 4.3). Since E has no proper reductions (by Corollary 2.4) then E is a complete intersection, and a) is proved. Now if $\mathcal{S}(E)$ is a domain then E is of linear type and b) follows by a). \square

We recall that E satisfies \widetilde{G}_s if $\mu(E_{\mathfrak{p}}) \leq \text{depth } R_{\mathfrak{p}} + e - 1$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $1 \leq \text{depth } R_{\mathfrak{p}} \leq s - 1$. Equivalently E satisfies \widetilde{G}_s if $\text{grade } F_i(E) \geq i - e + 2$ for $e \leq i \leq e + s - 2$. By [1, Proposition 5], the symmetric algebra of an ideal module over a domain with $\text{proj dim } E = 1$ and satisfying $\widetilde{G}_{\mu(E) - e + 1}$ is a domain. Hence we get the following consequence of Proposition 5.5.

Corollary 5.6. *Let R be a Noetherian local domain and E an ideal module having rank e with $\text{proj dim } E = 1$ and satisfying $\widetilde{G_{\mu(E)-e+1}}$. Then E is a complete intersection if and only if E is equimultiple.*

6. EXAMPLES OF IDEAL MODULES WITH SMALL REDUCTION NUMBER

In this section we observe that finite direct sums of ideals of grade ≥ 2 are ideal modules, and give examples of equimultiple and generically a complete intersection modules with small reduction number.

Proposition 6.1. *Let R be a Noetherian ring and $E = E_1 \oplus \cdots \oplus E_n$ with E_i finitely generated torsionfree R -modules having rank $e_i > 0$, $1 \leq i \leq n$, $n \geq 2$. Then E is an ideal module if and only if each summand E_i is an ideal module.*

Proof. Suppose that $E_i \subseteq G_i \simeq R^{e_i}$ and write $G = G_1 \oplus \cdots \oplus G_n$. Then G is a free R -module of rank $e = \sum_{i=1}^n e_i > 0$ and $E = E_1 \oplus \cdots \oplus E_n \subseteq G_1 \oplus \cdots \oplus G_n = G$. Since $G/E = (G_1 \oplus \cdots \oplus G_n)/(E_1 \oplus \cdots \oplus E_n) \simeq G_1/E_1 \oplus \cdots \oplus G_n/E_n$ then

$$\text{Supp } G/E = \text{Supp}(G_1/E_1 \oplus \cdots \oplus G_n/E_n) = \text{Supp } G_1/E_1 \cup \cdots \cup \text{Supp } G_n/E_n.$$

Therefore

$$\text{grade } G/E = \min_{1 \leq i \leq n} \{\text{grade } G_i/E_i\} \geq 2 \iff \text{grade } G_i/E_i \geq 2, 1 \leq i \leq n,$$

proving the equivalence. \square

We observe that a direct sum of ideals cannot be a complete intersection module.

Lemma 6.2. *Let R be a Noetherian ring and $E = E_1 \oplus \cdots \oplus E_n$ with E_i finitely generated R -modules having positive rank e_i , $1 \leq i \leq n$, $n \geq 2$. Then $F_e(E) = F_{e_1}(E_1) \cdots F_{e_n}(E_n)$, $e = \text{rank } E$. In particular $\text{grade } F_e(E) = \min_{1 \leq i \leq n} \{\text{grade } F_{e_i}(E_i)\}$ and $\text{ht } F_e(E) = \min_{1 \leq i \leq n} \{\text{ht } F_{e_i}(E_i)\}$.*

Proof. Since E_i has rank $e_i > 0$, then $F_k(E_i) = (0)$ for $k < e_i$. Now

$$F_e(E) = F_e(E_1 \oplus \cdots \oplus E_n) = \sum_{j_1 + \cdots + j_n = e} F_{j_1}(E_1) \cdots F_{j_n}(E_n) = F_{e_1}(E_1) \cdots F_{e_n}(E_n).$$

The other assertions follow. \square

Proposition 6.3. *Let (R, \mathfrak{m}, k) be a Noetherian local ring and $E = I_1 \oplus \cdots \oplus I_e$ with I_i R -ideals satisfying $\text{grade } I_i \geq 2$ for all $1 \leq i \leq e$, $e \geq 2$. Then E is not a complete intersection.*

Proof. By Proposition 6.1, E is an ideal module. Hence $\text{ht } F_e(E) \geq \text{grade } F_e(E) \geq 2$. Suppose that $\text{ht } F_e(E) = h \geq 2$. Whence $\text{ht } I_i \geq h$ for all i (by Lemma 6.2), and so $\mu(I_i) \geq h$ for all i . It follows that

$$\mu(E) - e + 1 = \sum_{i=1}^e \mu(I_i) - e + 1 \geq e h - e + 1 \geq 2h - 1 > h = \text{ht } F_e(E)$$

and E is not complete intersection. \square

Suppose that $E = I \oplus \cdots \oplus I = I^{\oplus e} \subseteq R^e$, $e \geq 2$, with I an R -ideal. For any $k \in \mathbb{N}_0$,

$$\mathcal{R}(I^{\oplus e})_k = (I^k)^{\oplus \binom{k+e-1}{e}} \simeq \mathcal{R}(\mathbf{I}_e)_k,$$

where $\mathcal{R}(\mathbf{I}_e)$ abbreviates the multi-Rees algebra $\mathcal{R}(I, \dots, I) = R[It_1, \dots, It_e]$. Suppose that $r(I) \leq 1$ and let J be a minimal reduction of I with $r_J(I) \leq 1$. Write $V = J \oplus \cdots \oplus J = J^{\oplus e} \subseteq E$. Then $I^2 = JI$ and we have

$$E^2 = (I^2)^{\oplus \binom{e+1}{2}} = (JI)^{\oplus \frac{(e+1)e}{2}} = V \cdot E.$$

Therefore V is a reduction of E with $r_V(E) \leq 1$. However, in general, V is not a minimal reduction of E . In the case where E is equimultiple with $\text{ht } I = 2$ we are able to construct a minimal reduction U of E with $r_U(E) \leq 1$.

Lemma 6.4. *Let (R, \mathfrak{m}) be a Noetherian local ring with infinite residue field. Then, for each $n \geq 2$ there exist $\alpha_1, \dots, \alpha_n \in R$ such that $\alpha_i - \alpha_j$ is a unit for all $1 \leq i < j \leq n$.*

Proof. Let $n \geq 2$. Since $k = R/\mathfrak{m}$ is infinite, there exist $\alpha_1, \dots, \alpha_n \in R$ such that $\alpha_i + \mathfrak{m} \neq \alpha_j + \mathfrak{m}$ for all $1 \leq i < j \leq n$. It follows that $\alpha_i - \alpha_j \in R \setminus \mathfrak{m} = R^*$ for all $1 \leq i < j \leq n$, proving the assertion. \square

Proposition 6.5. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with infinite residue field and $\dim R = d \geq 2$. Let I be an equimultiple ideal with $\text{ht } I = 2$ and $r(I) \leq 1$. Write $E = I \oplus \cdots \oplus I = I^{\oplus e}$, $e \geq 2$. Then:*

- a) $r(E) = 1$, $\ell(E) = e + 1$.
- b) E is equimultiple.

Proof. Let J be a minimal reduction of I with $r_J(I) \leq 1$. Then $I^2 = JI$. On the other hand, since $\ell(I) = \text{ht } I = 2$ then $J = \langle a, b \rangle$ for some $a, b \in I$. By the previous lemma there are $\alpha_1, \dots, \alpha_e \in R$ such that $\alpha_i - \alpha_j \in R^*$. Consider the family of elements

$$a_1 = a, b_1 = b, a_i = \alpha_i a + b, b_i = a \quad (i = 2, \dots, e).$$

We have, for each i, j

$$\begin{aligned} a &= (\alpha_i - \alpha_j)^{-1}(\alpha_i a + b) - (\alpha_i - \alpha_j)^{-1}(\alpha_j a + b), \\ b &= -\alpha_j(\alpha_i - \alpha_j)^{-1}(\alpha_i a + b) + (1 + \alpha_j(\alpha_i - \alpha_j)^{-1})(\alpha_j a + b). \end{aligned}$$

Therefore

$$J = \langle a_i, b_i \rangle = \langle a_i, a_j \rangle \quad (1 \leq i < j \leq e).$$

Thus,

$$I^2 = JI = a_i I + b_i I = a_i I + a_j I$$

for all $1 \leq i < j \leq e$. Consider the elements $x_i = a_i \epsilon_i$, $1 \leq i \leq e$, and $y = \sum_{i=1}^e b_i \epsilon_i$, where $(\epsilon_1, \dots, \epsilon_e)$ denotes the canonical basis of R^e , and consider the R -submodule U of E generated by x_1, \dots, x_e, y . We regard $\mathcal{R}(E)$ as a subalgebra of $R[t_1, \dots, t_e]$. So putting $I_i = I$ for $1 \leq i \leq e$

$$\begin{aligned} U \cdot E &= (Rx_1 + \cdots + Rx_e + Ry) \cdot (I_1 \oplus \cdots \oplus I_e) \\ &= (Ra_1 t_1 + \cdots + Ra_e t_e + R(b_1 t_1 + \cdots + b_e t_e)) \cdot (I_1 t_1 + \cdots + I_e t_e) \end{aligned}$$

$$= \sum_{i=1}^e a_i I_i t_i^2 + \left(\sum_{i=1}^e b_i t_i \right) I_1 t_1 + \cdots + \left(\sum_{i=1}^e b_i t_i \right) I_e t_e + \sum_{1 \leq i < j \leq e} I_i I_j t_i t_j,$$

and

$$\begin{aligned} E^2 &= I_1^2 \oplus \cdots \oplus I_e^2 \oplus \bigoplus_{1 \leq i < j \leq e} I_i I_j = \sum_{i=1}^e I_i^2 t_i^2 + \sum_{1 \leq i < j \leq e} I_i I_j t_i t_j \\ &= \sum_{i=1}^e a_i I_i t_i^2 + \sum_{i=1}^e b_i I_i t_i^2 + \sum_{1 \leq i < j \leq e} I_i I_j t_i t_j. \end{aligned}$$

Since

$$\sum_{i=1}^e b_i I_i t_i^2 \subseteq \left(\sum_{i=1}^e b_i t_i \right) I_1 t_1 + \cdots + \left(\sum_{i=1}^e b_i t_i \right) I_e t_e + \sum_{1 \leq i < j \leq e} I_i I_j t_i t_j$$

then $E^2 \subseteq U \cdot E \subseteq E^2$. Therefore U is a reduction of E with $r_U(E) \leq 1$. Moreover, $\mu(U) \leq e + 1 \leq \ell(E) \leq \mu(U)$, that is U is a minimal reduction of E . Thus $r(E) \leq r_U(E) \leq 1$ and $\ell(E) = \mu(U) = e + 1$.

On the other hand, E is an ideal module. Moreover, $\text{ht } F_e(E) = \text{ht } I = 2$ and $\ell(E) = e + 1$. Hence $\text{ad}(E) = \ell(E) - e + 1 - \text{ht } F_e(E) = 0$, and so E is equimultiple. Therefore, by Proposition 4.3 and by Proposition 6.3, $r(E) = 1$. \square

In particular, if I is a complete intersection ideal we obtain examples of equimultiple modules with reduction number equal to 1.

Corollary 6.6. *Let R be a Cohen-Macaulay local ring with infinite residue field and $\dim R = d \geq 2$. Let I be a complete intersection ideal with $\text{ht } I = 2$. Write $E = I \oplus \cdots \oplus I = I^{\oplus e}$, $e \geq 2$. Then E is equimultiple with $r(E) = 1$ and $\ell(E) = e + 1$.*

Corollary 6.7. *Let (R, \mathfrak{m}) be a regular local ring with infinite residue field and $\dim R = d = 2$. Let $E = \mathfrak{m} \oplus \cdots \oplus \mathfrak{m} = \mathfrak{m}^{\oplus e}$ with $e \geq 1$. Then E is equimultiple with $r(E) = 1$ and $\ell(E) = e + 1$.*

Next, we give examples of generically a complete intersection modules which are a direct sum of prime ideals.

Proposition 6.8. *Let R be a Cohen-Macaulay local ring with infinite residue field and $\dim R = d \geq 3$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_e$ be pairwise distinct prime ideals which are perfect of grade 2. Write $E = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_e$, $e \geq 2$. Then:*

- a) $\mu(E_{\mathfrak{p}}) \leq \text{ht } \mathfrak{p} + e - 1$ for every prime \mathfrak{p} with $1 \leq \text{ht } \mathfrak{p} \leq 2$.
- b) E is generically a complete intersection.
- c) E is not equimultiple.
- d) $\ell(E) \geq e + 2$, $\text{ad}(E) \geq 1$ with equalities if $d = 3$.
- e) If $d = 3$, $e = 2$ and $\mathfrak{p}_1, \mathfrak{p}_2$ are complete intersection then $r(E) = 0$.

Proof. a) Let $\mathfrak{p} \in \text{Spec}(R)$. If $\text{ht } \mathfrak{p} = 1$ then $\mathfrak{p} \neq \mathfrak{p}_i$ for all i . Hence $\mathfrak{p}_{i\mathfrak{p}} = \mathfrak{p}_i R_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all i . Thus

$$E_{\mathfrak{p}} = \mathfrak{p}_{1\mathfrak{p}} \oplus \cdots \oplus \mathfrak{p}_{e\mathfrak{p}} \simeq R_{\mathfrak{p}}^e,$$

and so $\mu(E_{\mathfrak{p}}) = e$. Now, suppose that $\text{ht } \mathfrak{p} = 2$. Then either $\mathfrak{p} \neq \mathfrak{p}_i$ for all i , so $\mu(E_{\mathfrak{p}}) = e \leq e + 1$, or $\mathfrak{p} = \mathfrak{p}_j$ for some j and $\mathfrak{p} \neq \mathfrak{p}_i$ for all $i \neq j$. In this case, $\mathfrak{p}_{j\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$. Moreover, since \mathfrak{p}_j is perfect of grade 2, then $\text{proj dim } R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} < \infty$. By [3, Theorem 2.2.7], $R_{\mathfrak{p}}$ is a regular local ring and so $\mathfrak{p}R_{\mathfrak{p}}$ is a complete intersection. Hence $\mu(\mathfrak{p}R_{\mathfrak{p}}) = \text{ht } \mathfrak{p}R_{\mathfrak{p}} = 2$. Therefore

$$E_{\mathfrak{p}} = \mathfrak{p}_{1\mathfrak{p}} \oplus \cdots \oplus \mathfrak{p}_{j\mathfrak{p}} \oplus \cdots \oplus \mathfrak{p}_{e\mathfrak{p}} \simeq R_{\mathfrak{p}} \oplus \cdots \oplus \mathfrak{p}R_{\mathfrak{p}} \oplus \cdots \oplus R_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^{e-1} \oplus \mathfrak{p}R_{\mathfrak{p}},$$

so that $\mu(E_{\mathfrak{p}}) = e - 1 + \mu(\mathfrak{p}R_{\mathfrak{p}}) = e + 1$.

b) By Corollary 6.1, E is an ideal module. Let $\mathfrak{p} \in \text{Min } R/F_e(E)$. We have $\mathfrak{p} \in V(F_e(E)) = V(\mathfrak{p}_1 \cdots \mathfrak{p}_e)$ and $\text{ht } \mathfrak{p} = \text{ht } F_e(E) = 2 = \text{ht } \mathfrak{p}_i$, for all i . Hence $\mathfrak{p} = \mathfrak{p}_j$ for some j and $\mathfrak{p} \neq \mathfrak{p}_i$ for $i \neq j$. Therefore, as above,

$$\mu(E_{\mathfrak{p}}) = e + 1 = \text{ht } F_e(E) + e - 1,$$

proving that E is generically a complete intersection.

c) follows by Proposition 5.3 and Proposition 6.3.

d) Since E is generically a complete intersection then $\ell(E_{\mathfrak{p}}) = \mu(E_{\mathfrak{p}}) = \text{ht } F_e(E) + e - 1 = e + 1$ for all $\mathfrak{p} \in \text{Min } R/F_e(E)$. Hence $\ell(E) \geq e + 1$ (by Corollary 2.3). If $\ell(E) = e + 1$ then E is equimultiple. It follows that $\ell(E) \geq e + 2$. Hence $\text{ad}(E) = \ell(E) - e + 1 - \text{ht } F_e(E) \geq 1$. Moreover, if $d = 3$ then $\ell(E) \leq d + e - 1 = e + 2$. Hence $\ell(E) = e + 2$ and so $\text{ad}(E) = 1$.

e) Since $d = 3$ and $\mathfrak{p}_1, \mathfrak{p}_2$ are c.i. ideals, then $\mu(E) = \mu(\mathfrak{p}_1) + \mu(\mathfrak{p}_2) = \text{ht } \mathfrak{p}_1 + \text{ht } \mathfrak{p}_2 = 4 = \ell(E)$. Hence E is a minimal reduction of itself, that is $r(E) = 0$. \square

We note that a direct sum of equimultiple modules is not always an equimultiple module. In the situation below, E is a direct sum of complete intersection ideals but E is not equimultiple (cf. Theorem 5.3 and Proposition 6.8).

Example 6.9. Let $R = k[[X_1, X_2, X_3]]$ with k an infinite field and write $E = \langle X_1, X_2 \rangle \oplus \langle X_1, X_3 \rangle$. Then E is generically a complete intersection with $\ell(E) = 4$, $\text{ad}(E) = 1$ and $r(E) = 0$.

Proof. R is a regular local ring with maximal ideal $\mathfrak{m} = \langle X_1, X_2, X_3 \rangle$, dimension $d = 3$ and $\mathfrak{p}_1 = \langle X_1, X_2 \rangle$, $\mathfrak{p}_2 = \langle X_1, X_3 \rangle$ are two distinct prime ideals of R with $\text{ht } \mathfrak{p}_i = 2 = \mu(\mathfrak{p}_i)$, $i = 1, 2$. The assertions then follow by Proposition 6.8. \square

7. OPEN CONDITIONS

Given a ring R , an R -module E and a property P it is very important to know if the subset $\{\mathfrak{p} \in \text{Spec}(R) \mid E_{\mathfrak{p}} \text{ satisfies } P\} \subset \text{Spec}(R)$ (the P locus) is open. For instance, for a finitely generated R -module E over a Noetherian ring R , it is well known that $U_n = \{\mathfrak{p} \in \text{Spec}(R) \mid \mu(E_{\mathfrak{p}}) \leq n\}$ and $U_F = \{\mathfrak{p} \in \text{Spec}(R) \mid E_{\mathfrak{p}} \text{ is free over } R_{\mathfrak{p}}\}$ are open subsets in $\text{Spec}(R)$.

Lemma 7.1. *Let R be a Noetherian ring, $0 \neq I$ an ideal in R and $\mathfrak{p} \in V(I)$. Then, there exists $\alpha \notin \mathfrak{p}$ such that $\text{ht } I R_{\alpha} = \text{ht } I_{\mathfrak{p}}$.*

Proof. Let

$$J = \bigcap_{\substack{\mathfrak{q} \in \text{Min } V(I) \\ \mathfrak{q} \not\subseteq \mathfrak{p}}} \mathfrak{q}.$$

If $J = \emptyset$ choose any $\alpha \notin \mathfrak{p}$. If $J \neq \emptyset$, then $0 \neq J$ is not contained in \mathfrak{p} and there exists $\alpha \in J$ such that $\alpha \notin \mathfrak{p}$. Hence, for any prime ideal $\mathfrak{q} \in \text{Min } V(I)$ such that $\alpha \notin \mathfrak{q}$, $\mathfrak{q} \subseteq \mathfrak{p}$. In both cases it is now clear that $\text{ht } IR_\alpha = \text{ht } I_\mathfrak{p}$. \square

The openness of the complete intersection locus of an ideal is well known. For ideal modules we have the following:

Theorem 7.2. *Let R be a Noetherian ring and $E \subsetneq G \simeq R^e$ an ideal module. Then $U_{ci} = \{\mathfrak{p} \in \text{Supp } G/E \mid E_\mathfrak{p} \text{ is a complete intersection}\}$ is a (possibly empty) open subset in $\text{Supp } G/E$.*

Proof. Let $\mathfrak{p} \in U_{ci}$ with $r = \mu(E_\mathfrak{p})$. Now, let $\alpha \notin \mathfrak{p}$ such that $\text{ht } F_e(E)R_\alpha = \text{ht } F_e(E_\mathfrak{p})$ (by Lemma 7.1). Then, $U_r \cap D(\alpha) = \{\mathfrak{q} \in D(\alpha) \mid \mu(E_\mathfrak{q}) \leq r\}$ is an open, non-empty set containing \mathfrak{p} , such that for any $\mathfrak{q} \in U_r \cap D(\alpha) \cap \text{Supp } G/E$, $E_\mathfrak{q}$ is an ideal module and

$$\begin{aligned} \mu(E_\mathfrak{q}) &\leq r = \mu(E_\mathfrak{p}) = e - 1 + \text{ht } F_e(E_\mathfrak{p}) \\ &= e - 1 + \text{ht } F_e(E)R_\alpha \leq e - 1 + \text{ht } F_e(E_\mathfrak{q}) \leq \mu(E_\mathfrak{q}). \end{aligned}$$

Hence,

$$\mu(E_\mathfrak{q}) = e - 1 + \text{ht } F_e(E_\mathfrak{q})$$

and $\mathfrak{q} \in U_{ci}$. \square

We note that U_{ci} may be an empty set (cf. Proposition 6.3).

Remark 7.3. Let R be a Noetherian ring and let $E \subseteq G \simeq R^e$, $e > 0$, be an R -module. Let $\mathfrak{p} \in \text{Spec}(R)$. Then, for every n , $(E^n)_\mathfrak{p} \simeq (E_\mathfrak{p})^n$. We simply write $E_\mathfrak{p}^n$ in any case.

The openness of the equimultiple locus of an ideal is proven, for instance, in [12]. Similarly, for ideal modules we get:

Theorem 7.4. *Let R be a Noetherian ring and $E \subsetneq G \simeq R^e$ an ideal module. Then $U_{eq} = \{\mathfrak{p} \in \text{Supp } G/E \mid E_\mathfrak{p} \text{ is equimultiple}\}$ is a non-empty open subset in $\text{Supp } G/E$.*

Proof. For any $\mathfrak{p} \in \text{Supp } G/E = V(F_e(E))$, $E_\mathfrak{p}$ is an ideal module and so $\text{ht } F_e(E) \leq \text{ht } F_e(E_\mathfrak{p}) \leq \ell(E_\mathfrak{p}) - e + 1 \leq \text{ht } \mathfrak{p}$. Now let $\mathfrak{p} \in V(F_e(E))$ minimal such that $\text{ht } \mathfrak{p} = \text{ht } F_e(E)$. Hence $\mathfrak{p} \in U_{eq}$ and so U_{eq} is non-empty.

Let $\mathfrak{p} \in U_{eq}$ with $s = \ell(E_\mathfrak{p})$ and $a_1, \dots, a_s \in E$ such that $\frac{a_1}{1}, \dots, \frac{a_s}{1}$ is a homogeneous system of parameters of $\mathcal{F}(E_\mathfrak{p})$, see Remark 2.2. Hence, for some r , $E_\mathfrak{p}^r = a_1 E_\mathfrak{p}^{r-r_1} + \dots + a_s E_\mathfrak{p}^{r-r_s}$. Therefore $\text{ann}_R(E^r/a_1 E^{r-r_1} + \dots + a_s E^{r-r_s}) \not\subseteq \mathfrak{p}$. Thus, by Lemma 7.1, we may choose $\alpha \in R \setminus \mathfrak{p}$ such that

$$\alpha E^r \subseteq a_1 E^{r-r_1} + \dots + a_s E^{r-r_s} \text{ and } \text{ht } F_e(E)R_\alpha = s - e + 1.$$

Therefore, for any $\mathfrak{q} \in D(\alpha) \cap \text{Supp } G/E$ we have

$$\alpha E_{\mathfrak{q}}^r = E_{\mathfrak{q}}^r$$

and so $E_{\mathfrak{q}}^r = a_1 E_{\mathfrak{q}}^{r-r_1} + \cdots + a_s E_{\mathfrak{q}}^{r-r_s}$, showing that $\ell(E_{\mathfrak{q}}) \leq s$. Thus we get

$$s \geq \ell(E_{\mathfrak{q}}) \geq \text{ht } F_e(E_{\mathfrak{q}}) + e - 1 \geq \text{ht } F_e(E)R_{\alpha} + e - 1 = s,$$

which proves that $D(\alpha) \cap \text{Supp } G/E \subseteq U_{eq}$. Therefore

$$U_{eq} = \bigcup_{\alpha} D(\alpha) \cap \text{Supp } G/E,$$

and so U_{eq} is an open subset in $\text{Supp } G/E$, as required. \square

8. THE REES POWERS OF AN IDEAL MODULE

In [5] we defined the n -th Rees power E^n of a finitely generated R -module $E \subset G \simeq R^e$ as the n -th graded piece of $\mathcal{R}(E)$

$$E^n := \mathcal{R}(E)_n \subset G^n \simeq R[t_1, \dots, t_e]_n \simeq R^{\binom{n+e-1}{e-1}},$$

in order to prove the Burch's inequality for modules.

Computing Rees powers of a module seems to be rather complicated even in the easiest cases.

Proposition 8.1. *Let (R, \mathfrak{m}) be a Noetherian local ring with $\dim R = d > 0$. Suppose that $E = F \oplus I$ where $F \simeq R^{e-1}$ and I is an R -ideal. Then*

- a) $E^n \simeq \bigoplus_{j=0}^n I^j R[t_1, \dots, t_{e-1}]_{n-j}$;
- b) $\text{depth } E^n = \min_{0 \leq j \leq n} \text{depth } I^j$;
- c) $\text{grade } \mathfrak{m}\mathcal{R}(E) = \text{grade } \mathfrak{m}\mathcal{R}(I)$.

Proof. Since F is a free module of rank $e-1$ then $\mathcal{R}(E) \simeq \mathcal{R}(I)[t_1, \dots, t_{e-1}]$ and a) follows. For b) note that $\text{depth } J = \text{depth } JG$ for any ideal J and any free module G . Hence, by a),

$$\text{depth } E^n = \min_{0 \leq j \leq n} \text{depth } I^j R[t_1, \dots, t_{e-1}]_{n-j} = \min_{0 \leq j \leq n} \text{depth } I^j.$$

Finally, using [5, Lemma 5.1]

$$\text{grade } \mathfrak{m}\mathcal{R}(E) = \inf_{n \geq 0} \text{depth } E^n = \inf_{n \geq 0} \text{depth } I^n = \text{grade } \mathfrak{m}\mathcal{R}(I),$$

proving c). \square

In the case where $E = I_1 \oplus \cdots \oplus I_e$ with $\text{grade } I_i > 0$, $i = 1, \dots, e$, then $\mathcal{R}(E) = \mathcal{R}(I_1 \oplus \cdots \oplus I_e)$ and so

$$E^n \simeq \bigoplus_{k_1 + \cdots + k_e = n} I^{k_1} \cdots I^{k_e}.$$

In particular, in the case where $I_1 = \cdots = I_e = I$ abbreviating $I \oplus \cdots \oplus I = I^{\oplus e}$ we get

$$E^n = I^n \oplus \cdots \oplus I^n = (I^n)^{\oplus \binom{n+e-1}{e-1}}.$$

In this case, $\text{depth } E^n = \text{depth } I^n$, for every $n \geq 1$.

Now we establish some basic properties about the quotients G^n/E^n for general finitely generated torsionfree R -modules. In fact, we prove that G^n/E^n has the same support, the same dimension and the same grade as G/E , and we apply this to ideal modules.

Proposition 8.2. *Let R be a Noetherian ring and let $E \subsetneq G \simeq R^e$, $e > 0$, be an R -module. Then, for every $n \geq 1$,*

- a) $\text{Supp } G^n/E^n = \text{Supp } G/E$;
- b) $\dim G^n/E^n = \dim G/E$;
- c) $\text{grade } G^n/E^n = \text{grade } G/E$;
- d) $\text{Min } G^n/E^n = \text{Min } G/E$;
- e) $G^n = E^n \iff G = E$.
- f) $1 \leq \text{depth } E^n \leq \text{depth } G^n/E^n + 1 \leq d - 1$ if R is Cohen-Macaulay.

Proof. Let $n \geq 1$.

a) The inclusion “ \subseteq ” is clear. On the other hand, suppose that $\mathfrak{p} \in \text{Spec}(R) \setminus \text{Supp } G^n/E^n$ then

$$E_{\mathfrak{p}}^n \subseteq E_{\mathfrak{p}} G_{\mathfrak{p}}^{n-1} \subseteq G_{\mathfrak{p}}^n = E_{\mathfrak{p}}^n,$$

and so $E_{\mathfrak{p}} G_{\mathfrak{p}}^{n-1} = G_{\mathfrak{p}}^n$, that is $E_{\mathfrak{p}}$ is a reduction of $G_{\mathfrak{p}}$. But $G_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module, hence $G_{\mathfrak{p}}$ has no proper reductions. Thus $E_{\mathfrak{p}} = G_{\mathfrak{p}}$ and so $\mathfrak{p} \notin \text{Supp } G/E$.

b) – e) are direct from a).

f) Since R is Cohen-Macaulay, $\text{depth } G^n = \text{depth } R = \dim R$. Now the inequality follows by the depth Lemma applied to the exact sequence $0 \rightarrow E^n \rightarrow G^n \rightarrow G^n/E^n \rightarrow 0$. \square

In the following E will be an ideal module. In this case we deduce some consequences of the result above. In fact, the equality $\text{Supp } G/E = \text{Supp } G^n/E^n$ implies that E^n is an ideal module, and so any result proved for G/E holds for G^n/E^n .

Corollary 8.3. *Let R be a Noetherian ring, E an ideal module having rank $e > 0$. Then, for every $n \geq 1$, E^n is an ideal module having rank $e_n = \binom{n+e-1}{e-1}$. Moreover, $V(F_{e_n}(E^n)) = V(F_e(E))$.*

Proof. The assertion is clear if E is free. Now suppose that E is not free and let $n \geq 1$. We first note that $E^n \neq 0$. Suppose that $E \subsetneq G \simeq R^e$. If $E^n = 0$ for some n , then

$$\text{Supp } G/E = \text{Supp } G^n/E^n = \text{Supp } G^n = \text{Spec}(R).$$

But $E_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^e \simeq G_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Ass } E$ and $\text{grade } G_{\mathfrak{p}}/E_{\mathfrak{p}} \geq 2$. Hence $E_{\mathfrak{p}} = G_{\mathfrak{p}}$ (cf. Lemma 3.4) – a contradiction. Therefore, $E^n \neq 0$. Moreover, by the previous result, $E^n \subsetneq G^n \simeq R^{\binom{n+e-1}{e-1}}$ with $\text{grade } G^n/E^n = \text{grade } G/E \geq 2$, and so E^n is an ideal module having rank e_n . \square

Corollary 8.4. *Let R be a Cohen-Macaulay local ring with $\dim R = d \geq 2$ and let $E \subsetneq G \simeq R^e$, be an ideal module having rank $e > 0$. If G/E is Cohen-Macaulay then $\text{depth } E^n \leq \text{depth } E$.*

Proof. Since G/E is Cohen-Macaulay,

$$\begin{aligned} \text{depth } E^n &= \text{depth } G^n/E^n + 1 \leq \dim G^n/E^n + 1 = \dim G/E + 1 \\ &= \text{depth } G/E + 1 = \text{depth } E, \end{aligned}$$

as asserted. \square

In the case where $\dim R = 2$, we have $\text{depth } E^n = 1$ and G^n/E^n is Cohen-Macaulay of dimension 0, for all $n \geq 1$, as in Corollary 3.16.

Corollary 8.5. *Let R be a Cohen-Macaulay local ring, $\dim R = 2$ and let $E \subsetneq G \simeq R^e$ with grade $G/E \geq 2$, be an ideal module over R having rank $e > 0$. Then, for every $n \geq 1$,*

- a) $\text{depth } E^n = 1$;
- b) $\dim G^n/E^n = \text{depth } G^n/E^n = 0$.

It is known that the analytic spread $\ell(I)$ of an ideal I over a Noetherian local ring satisfies the inequality

$$\ell(I) \leq \dim R - \inf_{n \geq 1} \text{depth } R/I^n$$

called the Burch's inequality. In [5] we proved that

$$\ell(E) \leq \dim R + e - 1 - \inf_{n \geq 1} \text{depth } G^n/E^n,$$

for a module $E \subset G \simeq R^e$. To do this we first proved that $\text{depth } G^n/E^n$ takes a constant value for large n . For equimultiple modules we are able to give an easy proof of this inequality.

Proposition 8.6. *Let R be a Noetherian local ring with $\dim R = d \geq 2$ and let $E \subsetneq G \simeq R^e$ be an equimultiple R -module having rank $e \geq 2$. Then*

$$\ell(E) \leq d + e - 1 - \inf_{n \geq 1} \text{depth } G^n/E^n.$$

Proof. We have, for every $n \geq 1$,

$$\text{depth } G^n/E^n \leq \dim G^n/E^n = \dim G/E \leq d - \text{ht } F_e(E).$$

Therefore, since E is equimultiple,

$$\inf_{n \geq 1} \text{depth } G^n/E^n \leq d - \text{ht } F_e(E) = d - \ell(E) + e - 1,$$

and the required inequality follows. \square

In the case where the Rees algebra $\mathcal{R}(E)$ is Cohen-Macaulay the Burch's inequality is an equality (cf. [5, Corollary 5.3]). In this case we obtain the following characterization for equimultiple modules.

Proposition 8.7. *Let R be a Cohen-Macaulay local ring, $\dim R = d > 0$ and let $E \subsetneq G \simeq R^e$ be an ideal module having rank $e > 0$ but not free. If $\mathcal{R}(E)$ is Cohen-Macaulay then the following are all equivalent:*

- a) E is equimultiple;
- b) $\text{depth } G^n/E^n = d - \text{ht } F_e(E)$ for all $n > 0$;

c) $\text{depth } G^n/E^n = d - \text{ht } F_e(E)$ for infinitely many n .

Proof. a) \Rightarrow b). Since E is equimultiple then

$$\text{ht } F_e(E) + e - 1 = \ell(E) = d + e - 1 - \inf_{n \geq 1} \text{depth } G^n/E^n,$$

and so $\text{depth } G^n/E^n \geq \inf_{n \geq 1} \text{depth } G^n/E^n = d - \text{ht } F_e(E) = \dim G/E$, for all $n > 0$.

b) \Rightarrow c) is immediate.

c) \Rightarrow a) follows by [5, Corollary 6.2]. \square

Corollary 8.8. *Let R be a Cohen-Macaulay local ring, $\dim R = d > 0$ and let $E \subsetneq G \simeq R^e$ be an ideal module having rank $e > 0$ but not free. If E is complete intersection then G^n/E^n are Cohen-Macaulay and $\text{Ass } G^n/E^n = \text{Ass } G/E = \text{Min } R/F_e(E)$, for all $n \geq 1$.*

Proof. In this case $\mathcal{R}(E)$ is Cohen-Macaulay and the assertions follow by Propositions 5.1, 8.2 and 8.7. \square

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